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THE SPECTRA AND COMMUTANTS OF SOME WEIGHTED COMPOSITION OPERATORS

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ABSTRACT. An operator T_{ug} on a Hilbert space H of functions on a set X defined by $T_{ug}(f) = u(f \circ g)$, where f is in H, $u: X \to \mathbf{C}$ and $g: X \to X$, is called a weighted composition operator. In this paper X is the set of integers and $H = L^2(\mathbf{Z}, \mu)$, where μ is a measure whose sigma-algebra is the power set of $\mathbf{Z}$. One distinguished space is $l^2 = L^2(\mathbf{Z}, \mu)$, where μ is counting measure. The most important results given here are the determination of the spectrum of T_{ug} on l^2 and a characterization of the commutant of T_g on $L^2(\mathbf{Z}, \mu)$. To obtain many of the results it was necessary to assume the function g to be one-to-one except on a finite subset of the integers.

INTRODUCTION

If H is a Hilbert space of complex valued functions on a set X, an operator A on H is called a composition operator if there is a function $g: X \to X$ such that $Af = f \circ g$ for all f in H. Composition operators that may be found in the literature are substitution operators, translation operators, and pointwise induced operators. Unilateral shifts, bilateral shifts, and translation on spaces $L^2(G)$, where G is a topological group, are examples of composition operators that have been studied extensively. An operator T on H is called a weighted composition operator if there is a complex valued function u on X and a function $g: X \to X$ such that $Tf = (u)(f \circ g)$ for all f in H. Weighted shifts on l^2 are examples of well-studied weighted composition operators (see Shields [25]). S. K. Parrott, in his thesis [22], studied multiplication by a function u on $H = L^2(X, v)$ and composition with an invertible measure preserving transformation g of $H = L^2(X, v)$, where v is a sigma-finite measure. In his paper [21], E. Nordgren summarized known information about composition operators on L^2 and H^2 spaces. An extensive reference list is also included in Nordgren's paper. After the appearance of Nordgren's paper other authors, such as Kumar, Singh, and Whitley, presented results about composition operators on L^2 and other authors, such as Cowen, Kamowtiz, Kumar, and Marshall presented results about composition operators on H^2.

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The definition of weighted composition operator given above is too general to hope to obtain many results. In the present work, study has been restricted to the case when X is the set of integers and $H = l^2$ is viewed as a Hilbert space of functions on the integers. A few papers that have appeared concerning weighted composition operators in this setting are those of Gupta, Kumar, and Singh [10], Komal and Singh [14], and Singh [28]. Gupta, Kumar, and Singh [10] discuss quasinormal composition operators on a weighted space l^2_p where the weights are square summable; Komal and Singh [14] give elementary properties of composition operators on l^p spaces; Singh [28] provides information about composition operators on weighted spaces $L^2(\lambda)$.

Throughout this work $\mathbf{Z}$, $\mathbf{N}$, $\mathbf{R}$, $\mathbf{C}$, and $\mathbf{T}$ denote the sets of integers, positive integers, real numbers, complex numbers and the set of complex numbers of modulus one, respectively. By the word operator we will mean a linear operator but not necessarily a bounded operator. If T is an operator then $\sigma(T)$, $\sigma_p(T)$, $\sigma_e(T)$, and $\{T\}'$ denote the spectrum, point spectrum, essential spectrum, and commutant of T. Also, $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ denote the norm and inner product on l^2 and, for each n in $\mathbf{Z}$, e_n denotes the function on $\mathbf{Z}$ that is one at n and zero otherwise. Note that $\{e_n\}_{n\in\mathbf{Z}}$ is an orthonormal basis for l^2.

1. Preliminaries

The formal definition of a weighted composition operator used in this work is stated here:

If T is a linear operator on l^2 defined by

$$Tf(n) = \begin{cases} u(n)(f\circ g) & \text{if } n \text{ is in } Y, \\ 0 & \text{otherwise} \end{cases}$$

for all f in l^2 and n in $\mathbf{Z}$, where Y is a subset of $\mathbf{Z}$, $g\colon Y \to \mathbf{Z}$ and $u\colon \mathbf{Z} \to \mathbf{C}$, then T is called a *weighted composition operator* on l^2. In this case we will denote T by T_{ug}, u will be called a *weight function*, and g will be called a *composition function*. If u is identically equal to one then the notation T_g may be used for T_{ug}.

If $n \geq 0$ then the nth-iterate of g is given by $g^{(n)} = g\circ g\circ\cdots\circ g$, g composed with itself n-times. If $n > 0$ and k is in $\mathbf{Z}$ then $g^{-n}(k) = \{j \text{ in } \mathbf{Z}\colon g^{(n)}(j) = k\}$. In the case when $g^{-n}(k)$ is a singleton, $g^{-n}(k) = \{j\}$, there will be no distinction made between j and $\{j\}$.

It will be helpful to find formulas for T_{ug} and T^*_{ug}. Let $f = \sum_{n\in\mathbf{Z}} f_n e_n$ be in l^2, Y be a subset of $\mathbf{Z}$, $g\colon Y \to \mathbf{Z}$ and $u\colon \mathbf{Z}\to\mathbf{C}$. Then

$$T_{ug}(f) = \sum_{n\in Y} f_{g(n)} u(n) e_n$$

and

$$T^*_{ug}(f) = \sum_{n\in \text{Image } g}\ \sum_{k\in g^{-1}(n)} f_k \overline{u(k)} e_n;$$

the sums given here are formal sums that may not converge. In the case $f = e_n$ for some n in $\mathbf{Z}$,

$$T_{ug}e_n = \begin{cases} \sum_{k\in g^{-1}(n)} u(k)e_k & \text{if } n \text{ is in Image } g, \\ 0 & \text{otherwise}, \end{cases}$$

$$T^*_{ug}e_n = \begin{cases} \overline{u(n)}e_{g(n)} & \text{if } n \text{ is in } Y, \\ 0 & \text{otherwise.} \end{cases}$$

Let Y be a subset of $\mathbf{Z}$ and $g\colon Y \to \mathbf{Z}$ be a function. For n in $\mathbf{Z}$ the set $[n]_g = \{k \text{ in } \mathbf{Z}\colon \text{ there exist } j \geq 0 \text{ and } t \geq 0 \text{ such that } g^{(j)}(k) = g^{(t)}(n)\}$ is called the *orbit of* g containing n. If k is in $\mathbf{Z}$ and $g^{(j)}(k) = k$ for some $j \geq 1$ then the *cycle of* g containing k is the set $C = \{n \text{ in } \mathbf{Z}\colon g^{(t)}(k) = n$ for some $t \geq 0\}$; the *length of the cycle* C is the least integer $j \geq 1$ such that $g^{(j)}(k) = k$. Note that if C is a cycle of g of length j and if n is in C then $g^{(j)}(n) = n$.

Let Y be a subset of $\mathbf{Z}$ and $g\colon Y \to \mathbf{Z}$. Define the function $[g]\colon (Y \cup \text{Image } g) \to \mathbf{Z}$ by

(1) $[g](n) = 0$ if n is in a cycle of g,
(2) $[g](n_F) = 0$, where n_F is a fixed element of each orbit F of g not containing a cycle,
(3) $[g](n) + 1 = [g](g(n))$ if n is in Y and n is not in a cycle of g.

For the purposes of this paper the choice of the integers n_F is not important since any other choice can be obtained under a reindexing of the integers. The *branch of* g containing n is a maximal subset G of $[n]_g$ such that n is in G, $[g]$ restricted to $G \cap Y$ is one-to-one, and n in $G \cap Y$ implies $g(n)$ is in G if n is not a cycle of g. The set of all branches of g will be denoted by $\operatorname{br} g$. We will now exploit the fact that a branch of g is linearly ordered by the function $[g]$. For a branch b of g let $\operatorname{nl}(b) = \inf[g](b)$ be the *negative length of* b and let $\operatorname{pl}(b) = \sup[g](b)$ be the *positive length of* b. A branch b of g gives rise to a function from $[g](b)$ into b, this function will also be denoted by b and is given by $b(n) = [g]^{-1}(n) \cap b$ for each n in $[g](b)$.

Example 1. Let u be a weight function on $\mathbf{N}$ and let g and h be composition functions given by $g(n) = n - 1$ if $n \geq 1$ and $h(n) = n + 1$ if $n \geq 0$ (under a bijection from $\mathbf{N}$ onto $\mathbf{Z}$, g and h can be thought of as functions on $\mathbf{Z}$). The operator T_{ug} is a weighted forward shift and T_{uh} is a weighted backward shift. □

One way to visualize weighted composition operators is to use "graphs". In Example 1 the graph

$$\cdots \to 3 \to 2 \to 1 \to 0$$

indicates the action of g on $\mathbf{N}$. The graph

$$e_0 \to e_1 \to e_2 \to e_3 \to \cdots$$

indicates the action of T_{ug} on the basis $\{e_n\}_{n\geq 0}$. Note that an arrow denotes one basis element being sent into a multiple of the next.

Example 2. If $g\colon \mathbf{Z}\to\mathbf{Z}$ is defined by $g(n)=n+1$ when $n\geq 0$ and $g(n)=n+2$ when $n<0$ then g is visualized by the graph

$$\begin{array}{l} \cdots\to -4\to -2\to 0\to 1\to 2\to 3\to\cdots \\ \qquad\qquad\qquad\quad\;\;\nearrow \\ \cdots\to -5\to -3\to -1. \end{array}$$

The function g has two branches that may be visualized as

$$\cdots\to -4\to -2\to 0\to 1\to 2\to 3\to\cdots$$

and

$$\cdots\to -5\to -3\to -1\to 1\to 2\to 3\to\cdots.$$

The operator T_g can be visualized by the graph

$$\begin{array}{lll} \cdots\to e_2\to e_1\to & e_0\to e_{-2}\to\cdots \\ \qquad\qquad\quad\;\searrow & \\ & e_{-1}\to e_{-3}\to\cdots \end{array}$$

where the arrows from e_1 imply e_1 is sent into a linear combination of e_0 and e_{-1}. The operator T_g^* can be visualized by

$$\begin{array}{l} \cdots\to e_{-2}\to e_0\to e_1\to e_2\to\cdots \\ \qquad\qquad\qquad\nearrow \\ \cdots\to e_{-3}\to e_{-1}. \qquad \square \end{array}$$

Example 3. Let $f(n)=(n-1)(n-2)(n-3)$ for n in $\mathbf{Z}$, $Y=\{k \text{ in } \mathbf{Z}\colon k=f^{(j)}(i)$ for some $j\geq 0$ and $i=1,2,3\}$, and g be f restricted to Y. The graph for T_g is

$$\begin{array}{ll} \cdots\to e_{-504}\to e_{-6}\to e_0\to & e_1 \\ \qquad\qquad\qquad\qquad\quad\;\searrow & \\ & e_2 \\ \qquad\qquad\qquad\qquad\quad\;\searrow & \\ & e_3. \quad \square \end{array}$$

Example 4. This example is motivated by a paper of C. C. Cowen [7]. Let D be the unit disc in $\mathbf{C}$ and $g\colon D\to D$ be a nonconstant analytic function. If z is a point in D then $F=\{b\colon g^{(k)}(b)=z$ for some $k>0\}$ is a countable set. Index F by the negative integers to obtain $F=\{z_k\}_{k<0}$ and for $k>-1$ let $z_k=g^{(k)}(z)$. The function g induces $g^{\sim}\colon\mathbf{Z}\to\mathbf{Z}$ by $g^{\sim}(k)=j$ if $g(z_k)=z_j$. If g has a Denjoy-Wolff point a in $\mathbf{T}$, $g'(a)=1$ and $G=\{f$ in $H^2(D)\colon f(z_k)=0$ for all k in $\mathbf{Z}\}$ then $G^{\perp}$ is a nontrivial invariant subspace for T_g^* (see Cowen [7]). On $G^\perp$ the operator T_g^* is similar to a weighted composition operator $T_{ug\sim}$ on l^2. □

To conclude this section we point out that weighted composition operators are related to operator weighted shifts. Let $g: \mathbf{Z} \to \mathbf{Z}$, $Y \cup \text{Image}\, g = \mathbf{Z}$, and u be a nonzero weight function. For each n in $\mathbf{Z}$ define H_n to be the closed span of $\{e_k : [g](k) = n\}$. The operator T_{ug} maps H_n into H_{n-1} for each n in $\mathbf{Z}$.

2. Spectrum

In this section we want to find the point spectrum, essential spectrum, and spectrum of a weighted composition operator. We will restrict ourselves to the case when a composition function has finitely many branches. For this case the point spectrum, essential spectrum, and spectrum of a weighted composition operator are found in Theorems 2.1, 2.2, and 2.3. We will need only consider composition functions with one orbit, since an orbit induces a reducing subspace to which the restriction of the weighted composition operator is again a weighted composition operator. Also, we will assume that $\mathbf{Z}$ is equal to the orbit of the composition function, since a weighted composition operator restricted to the basis elements associated with integers not in an orbit of the composition function is a zero operator.

Let Y be a subset of $\mathbf{Z}$, $g: Y \to \mathbf{Z}$ have one orbit and finitely many branches, $\mathbf{Z} = (\text{Image} g) \cup Y$, and u be a nonzero weight function. The notation given here will make many of the following statements less cumbersome. If b is a branch of g and j is in $[g](b)$ then define

$$(1)\qquad U_b = \begin{cases} \lim_{n\geq 1} \sup |u(b(j-n)) \cdots u(b(j-1))|^{1/n} & \text{if } \mathrm{nl}(b) = -\infty, \\ 0 & \text{otherwise}, \end{cases}$$

$$(2)\qquad L_b = \begin{cases} \lim_{n\geq 1} \inf |u(b(j-n)) \cdots u(b(j-1))|^{1/n} & \text{if } \mathrm{nl}(b) = -\infty, \\ 0 & \text{otherwise}, \end{cases}$$

$$(3)\qquad L = \begin{cases} \lim_{n\geq 1} \inf |u(b(j+1)) \cdots u(b(j+n))|^{1/n} & \text{if } \mathrm{pl}(b) = +\infty, \\ 0 & \text{otherwise}, \end{cases}$$

$$(4)\qquad U = \begin{cases} \lim_{n\geq 1} \sup |u(b(j+1)) \cdots u(b(j+n))|^{1/n} & \text{if } \mathrm{pl}(b) = +\infty, \\ 0 & \text{otherwise.} \end{cases}$$

The numbers U_b and L_b do not depend on the choice of j in $[g](b)$. The numbers L and U do not depend on b or j in $[g](b)$.

In Theorem 2.1 we find that the point spectrum of T_{ug} and T_{ug}^*, when g does not have a cycle, is either a disk, an annulus, or a disk union an annulus. The particular structure of the spectrum depends on the values U_b, L_b, U and L for all branches b of g. In Theorem 2.2 we find that the point spectrum of T_{ug}, when g has a cycle, is a finite set of points and that the point spectrum of T_{ug}^* is a finite set of points union a disk. When g does not have a cycle the spectrum of T_{ug} has circular symmetry since T_{ug} is unitarily equivalent to $T_{|u|g}$ (Carlson [1]).

The results obtained here were foreshadowed by the known structure of the spectrum of a weighted shift on l^2 (Ridge [23], Shields [25]). A weighted shift is merely a weighted composition operator with one branch b where the spectrum is either an annulus or a disk determined by the values U_b and U. The question answered here is 'what happens to the spectrum when weighted shifts are pasted together'. With the assumption that the composition operator has only a finite number of branches the results on weighed shifts can be generalized to weighted composition operators by analyzing the point spectrum of the operator and its adjoint.

Theorem 2.1. *Let Y be a subset of $\mathbf{Z}$, $g\colon Y \to \mathbf{Z}$ have one orbit, finitely many branches, and no cycle, $\mathbf{Z} = Y \cup \operatorname{Image} g$, u be a nonzero weight function, and T_{ug} be a bounded operator. Then*

$$B_1 \cup (\textit{Interior of } K_1) \subset \sigma_p(T_{ug}) \subset B_1 \cup K_1$$

and

$$B_2 \cup \textit{Interior of } (K_2 \cup K_3) \subset \sigma_p(T^*_{ug}) \subset B_2 \cup K_2 \cup K_3,$$

where $K_1 = \{z \text{ in } \mathbf{C}\colon U_a \leq |z| \leq L \text{ for all } a \text{ in } \operatorname{br} g\}$, $K_2 = \{z \text{ in } \mathbf{C}\colon U \leq |z| \leq L_a \text{ for some } a \text{ in } \operatorname{br} g\}$, $K_3 = \{z \text{ in } \mathbf{C}\colon |z| \leq L_a \leq L_b \text{ for some } a \text{ and } b \text{ in } \operatorname{br} g \text{ such that } a \neq b\}$, B_1 is $\{0\}$ if $\mathbf{Z}\backslash \operatorname{Image} g$ is not empty and is empty otherwise, and B_2 is $\{0\}$ if either $\mathbf{Z}\backslash Y$ is not empty or g is not one-to-one and is empty otherwise.

Proof. Let $f = \sum_{n\in\mathbf{Z}} f_n e_n$ be in l^2 and z be in $\mathbf{C}$.

$$(2.1)\qquad \begin{aligned}(T_{ug} - zI)f &= \sum_{n\in Y} f_{g(n)}u(n)e_n - \sum_{n\in\mathbf{Z}} zf_n e_n \\ &= \sum_{n\in Y}(f_{g(n)}u(n) - zf_n)e_n + \sum_{n\in\mathbf{Z}\backslash Y} zf_n e_n\end{aligned}$$

and

$$(2.2)\qquad \begin{aligned}(T^*_{ug} - zI)f &= \sum_{n\in\operatorname{Image} g}\ \sum_{k\in g^{-1}(n)} f_k\overline{u(k)}e_n - \sum_{n\in\mathbf{Z}} zf_n e_n \\ &= \sum_{n\in\operatorname{Image} g}\left(-zf_n + \sum_{k\in g^{-1}(n)} f_k\overline{u(k)}\right)e_n - \sum_{n\in\mathbf{Z}\backslash \operatorname{Image} g} zf_n e_n.\end{aligned}$$

Let $j = \min[g](\bigcap_{a\in\operatorname{br} g} a)$ when g has more than one branch and let j be an element of $[g](\mathbf{Z})$ when g has only one branch.

Suppose zero is in the point spectrum of T_{ug}; then there is an $f = \sum_{n\in\mathbf{Z}} f_n e_n$ in l^2 such that f is not zero and $T_{ug}f = 0$. So (2.1) implies $f_{g(n)}u(n) = 0$ for all n in Y. Thus, $f_m = 0$ for all m in $\operatorname{Image} g$, implying $\mathbf{Z}\backslash \operatorname{Image} g$ is not empty.

If $\mathbf{Z}\backslash \operatorname{Image} g$ is not empty let n be in $\mathbf{Z}\backslash \operatorname{Image} g$; then $T_{ug}e_n = 0$. Thus, if $\mathbf{Z}\backslash \operatorname{Image} g$ is not empty then 0 is in the point spectrum of T_{ug}. Therefore 0 is in the point spectrum of T_{ug} if and only if $\mathbf{Z}\backslash \operatorname{Image} g$ is not empty. Hence, $B_1 \subset \sigma_p(T_{ug})$.

Let z in $\mathbf{C}$ be nonzero and $f = \sum_{n\in\mathbf{Z}} f_n e_n$ in l^2 not be zero such that $(T_{ug} - zI)f = 0$. From (2.1) we have $f_{g(n)}u(n) - zf_n = 0$ for all n in Y and $f_n = 0$ for all n in $\mathbf{Z}\backslash Y$. Since $z \neq 0$, u is nonzero, and $\mathbf{Z} = \operatorname{Image} g \cup Y$, $f_n = 0$ if and only if $f_{g(n)} = 0$ for all n in Y. So, either $f_n = 0$ for all n in $\mathbf{Z}$ or $f_n \neq 0$ for all n in $\mathbf{Z}$. Hence, $\mathbf{Z}\backslash Y$ is empty since f is nonzero. So, $\operatorname{pl}(a) = +\infty$ for all a in $\operatorname{br} g$. Recall that for a branch a of g we denote an element of a by $a(n)$ for some n in $\mathbf{Z}$, so for each branch a of g we have $f_{a(n+1)}u(n) - zf_{a(n)} = 0$ for n in $[g](a)$. Now, for a in $\operatorname{br} g$ and $m \geq \operatorname{nl}(a)$, if $0 \leq n$ then

$$f_{a(n+m+1)} = z^{n+1}(u(a(m))\cdots u(a(n+m)))^{-1} f_{a(m)} \neq 0,$$

and if $\operatorname{nl}(a) \leq m - n \leq m$ then

$$f_{a(m-n)} = z^{-n} u(a(m-n))\cdots u(a(m-1)) f_{a(m)} \neq 0.$$

For a in $\operatorname{br} g$ and $m \geq \operatorname{nl}(a)$ the sequences

$$\{z^{n+1}(u(a(m))\cdots u(a(n+m)))^{-1}\}_{n\geq 0}$$

and

$$\{z^{j-m} u(a(j))\cdots u(a(m-1))\}_{\operatorname{nl}(a)\leq j\leq m+1}$$

are square summable. Thus, the nonzero elements of the point spectrum of T_{ug} are contained in K_1.

Let z be in interior of K_1 and $z \neq 0$. If $\mathbf{Z}\backslash Y$ is not empty then $L = 0$ and Interior of K_1 is empty. Thus, $\mathbf{Z}\backslash Y$ is empty and $\operatorname{pl}(a) = +\infty$ for each a in $\operatorname{br} g$. We wish to find f in l^2 such that $(T_{ug} - zI)f = 0$. Let $f = \sum_{n\in\mathbf{Z}} f_n e_n$ where $f_n = 1$ for $[g](n) = j$, $f_n = u(n)\cdots u(g^{(j-m-1)}(n))z^{m-j}$ for $m = [g](n) < j$, and $f_n = (u(g^{(m-j-1)}(i))\cdots u(i))^{-1} z^{m-j}$ for $[g](i) = j$ and $m = [g](n) > j$. Clearly, $f \neq 0$. For n in $\mathbf{Z}$ such that $[g](n) = m$, if a is a branch containing n then $g^{(t)}(n) = a(m+t)$ for $t \geq 0$. Thus, for n in $\mathbf{Z}$ and n contained in the branch a, if $m = [g](n) < j$ then

$$f_n = z^{m-j} u(a(m))\cdots u(a(j-1)),$$

and if $m = [g](n) > j$ then $f_n = z^{m-j}(u(a(j))\cdots u(a(m-1)))^{-1}$. The sequence $\{f_n\}_{n\in\mathbf{Z}}$ is square summable since g has finitely many branches and z

is in K_1. Hence f is in l^2. Now

$$\begin{aligned}(T_{ug} - zI)f &= \sum_{n\in Y}(f_{g(n)}u(n) - zf_n)e_n \\ &= \sum_{\substack{n\in Y \\ m=[g](n)<j-1}} (u(n)u(g(n))\cdots u(g^{(j-m-1)}(n))z^{m-j+1} \\ &\qquad\qquad - zu(n)\cdots u(g^{(j-m-1)}(n))z^{m-j}) \\ &\quad + \sum_{\substack{n\in Y \\ m=[g](n)=j-1}} (u(n) - zu(n)z^{-1}) + (u(i)(u(i))^{-1}z - z) \\ &\quad + \sum_{\substack{n\in Y \\ m=[g](n)>j}} (u(i)\cdots u(g^{(m-j)}(i)))^{-1}u(g^{(m-j)}(i))z^{m+j+1} \\ &\qquad\qquad - z(u(i)\cdots u(g^{(m-j-1)}(i)))^{-1}z^{m-j}) \\ &= 0,\end{aligned}$$

where $[g](i) = j$. Thus, f is an eigenvector for T_{ug} with eigenvalue z. The set Interior of K_1 is contained in the point spectrum of T_{ug}.

Suppose that zero is in the point spectrum of T_{ug}^*. There is an $f = \sum_{n\in\mathbf{Z}} f_n e_n$ in l^2 such that $f \neq 0$ and $T_{ug}^* f = 0$. By (2.2) $\sum_{k\in g^{-1}(n)} f_k u(k) = 0$ for all n in Image g. If g is one-to-one and $\mathbf{Z} = Y$ then $f_k\overline{u(k)} = 0$ for all k in $\mathbf{Z} = Y$. Thus, g is not one-to-one or $\mathbf{Z}\backslash Y$ is not empty.

Suppose g is not one-to-one. There are k_1 and k_2 in Y such that $k_1 \neq k_2$ and $g(k_1) = g(k_2)$. The function $f = \overline{u(k_2)}e_{k_1} - \overline{u(k_1)}e_{k_2}$ is in l^2 and $T_{ug}^* f = 0$.

Suppose $\mathbf{Z}\backslash Y$ is not empty. There is an n in $\mathbf{Z}$ such that n is not in Y. Now $T_{ug}^* e_n = 0$ and zero is in the point spectrum of T_{ug}^*. Thus, $B_2 \subset \sigma_p(T_{ug}^*)$.

Let z in $\mathbf{C}$ be nonzero and f in l^2 be such that $f \neq 0$ and $(T_{ug}^* - zI)f = 0$. By (2.2), $zf_n - \sum_{k\in g^{-1}(n)} f_k\overline{u(k)} = 0$ for all n in Image g and $zf_n = 0$ for all n in $\mathbf{Z}\backslash$ Image g. Either there is a branch a of g and $K > 0$ such that $\mathrm{nl}(a) = -\infty$, $\mathrm{pl}(a) = +\infty$,

$$f_{a(-n-K)} = z^n(\overline{u(a(-n-K))\cdots u(a(-K-1))})^{-1} f_{a(-K)} \neq 0$$

and

$$f_{a(n+K)} = z^{-n}\overline{u(a(K))\cdots u(a(n+K-1))} f_{a(-K)} \neq 0$$

for all $n \geq K$ or there are two distinct branches a and b of g and $K > 0$ such that $\mathrm{nl}(a) = -\infty$, $\mathrm{nl}(b) = -\infty$,

$$0 \neq f_{a(-n-K)} = z^n(\overline{u(a(-n-K))\cdots u(a(-K-1))})^{-1} f_{a(-K)},$$

and

$$0 \neq f_{b(-n-K)} = z^n(\overline{u(b(-n-K))\cdots u(b(-K-1))})^{-1} f_{b(-K)}$$

for all $n > 0$.

So, either there is a branch a such that $\mathrm{pl}(a) = +\infty$, $\mathrm{nl}(a) = -\infty$, and $\{z^{-n}u(a(j))\cdots u(a(n+j-1))\}_{n\geq 1}$ and $\{z^n(\overline{u(a(-n+j))\cdots u(a(-1+j))})^{-1}\}_{n\geq 1}$ are square summable or there are two distinct branches a and b such that $\mathrm{nl}(a) = -\infty$, $\mathrm{nl}(b) = -\infty$, and

$$\{z^n(\overline{u(a(-n+j))\cdots u(a(-1+j))})^{-1}\}_{n\geq 1}$$

and

$$\{z^n(\overline{u(b(j))\cdots u(b(n+j-1))})^{-1}\}_{n\geq 1}$$

are square summable. Thus, z is contained in $K_2 \cup K_3$.

Let z be in Interior of K_2 (note that $z \neq 0$). There is a branch a of g such that $U < |z| < L_a$. We will now find an f in l^2 such that $(T^*_{ug} - zI)f = 0$. Define $f = \sum_{n\in\mathbf{Z}} f_{a(n)}e_{a(n)}$, where

$$f_{a(j)} = 1, \quad f_{a(n+j)} = z^{-n}\overline{u(a(j))\cdots u(a(n+j-1))} \quad \text{for } n \geq 1,$$

and

$$f_{a(j-n)} = z^n(u(a(j-n))\cdots u(a(j-1)))^{-1} \quad \text{for } n \geq 1.$$

Clearly, $f \neq 0$ and f is in l^2. Furthermore,

$$(T^*_{ug} - zI)f = \sum_{n\in\mathbf{Z}}(-zf_{a(n)} + f_{a(n-1)}\overline{u(a(n-1))})e_{a(n)} = 0.$$

Thus, f is an eigenvector of T^*_{ug} for the eigenvalue z.

Let z be in Interior of K_3 such that $z \neq 0$. There are two distinct branches a and b of g such that $z < L_a \leq L_b$. Define $f = \sum_{n\leq 1} f_{a(n)}e_{a(n)} + f_{b(n)}e_{b(n)}$ where $1 + i = \min[g](a \cap b)$, $f_{a(i)} = \overline{u(b(i))}$, $f_{b(i)} = -\overline{u(a(i))}$, and $f_{a(i-n)} = \overline{u(b(i))}z^n(\overline{u(a(i-n))\cdots u(a(i-1))})^{-1}$ for $n \geq 1$. The function f is in l^2, f is not zero, and $(T^*_{ug} - zI)f = 0$. Thus, f is an eigenvector of T^*_{ug} for the eigenvalue z. □

Theorem 2.2. *Let $g\colon \mathbf{Z} \to \mathbf{Z}$ have one orbit, finitely many branches, and exactly one cycle C of length c, let u be a nonzero weight function, and let T_{ug} be a bounded operator. Then $B \cup K_1 \subset \sigma_p(T_{ug}) \subset K_2 \cup B$ and $K_3 \cup \{0\} \subset \sigma_p(T^*_{ug}) \subset K_4 \cup \{0\}$, where $K_1 = \{z$ in $\mathbf{C}\colon z^c = u(n)\cdots u(g^{(c-1)}(n))$ for n in C and $|z| > U_a$ for all a in $\mathrm{br}\, g\}$, $K_2 = \{z$ in $\mathbf{C}\colon z^c = u(n)\cdots u(g^{(c-1)}(n))$ for n in C and $|z| \geq U_a$ for all a in $\mathrm{br}\, g\}$, $K_3 = \{z \in \mathbf{C}\colon z^c = u(n)\cdots u(g^{(c-1)}(n))$ for n in C or $|z| < L_a \leq L_b$ for some a and b in $\mathrm{br}\, g$ such that $a \neq b\}$, $K_4 = \{z$ in $\mathbf{C}\colon z^c = u(n)\cdots u(g^{(c-1)}(n))$ for n in C or $|z| \leq L_a \leq L_b$ for some a and b in $\mathrm{br}\, g$ such that $a \neq b\}$, and B is $\{0\}$ when $\mathbf{Z}\backslash \mathrm{Image}\, g$ is not empty or B is empty otherwise.*

Proof. If $f = \sum_{n\in\mathbf{Z}} f_n e_n$ is in l^2 and z is in C then the formulas (2.1) and (2.2) still hold.

Suppose $f = \sum_{n\in\mathbf{Z}} f_n e_n$ in l^2 is not zero and $T_{ug} f = 0$. By (2.1), $f_n \neq 0$ for some n in $\mathbf{Z}\backslash \operatorname{Image} g$. Thus, if zero is an eigenvalue for T_{ug} then $\mathbf{Z}\backslash \operatorname{Image} g$ is not empty.

Suppose there is an n in $\mathbf{Z}\backslash \operatorname{Image} g$. Then $T_{ug} e_n = 0$. Thus, if $\mathbf{Z}\backslash \operatorname{Image} g$ is not empty then zero is an eigenvalue of T_{ug}. Therefore zero is in the point spectrum of T_{ug} if and only if $\mathbf{Z}\backslash \operatorname{Image} g$ is not empty.

Suppose z in $\mathbf{C}$ is not zero and $f = \sum_{n\in\mathbf{Z}} f_n e_n$ in l^2 is not zero such that $(T_{ug} - zI)f = 0$. From (2.1) we have $f_{g(n)} u(n) - z f_n = 0$ for all n in Y and $f_n = 0$ for all n not in Y. So

$$f_{g^{(k)}(n)} = z^k (u(n) \cdots u(g^{(k-1)}(n)))^{-1} f_n$$

for n in C and $k \geq 1$ and

$$f_{a(-k)} = z^{-k} u(a(-k)) \cdots u(a(-1)) f_{a(0)}$$

for all branches a of g and $1 \leq k \leq |\operatorname{nl}(a)|$. Hence, $z^c = u(n) \cdots u(g^{(c-1)}(n))$ for n in C and the sequence $\{z^{-k} u(a(-k)) \cdots u(a(-1))\}_{k=1}^{|\operatorname{nl}(a)|}$ is square summable for each a in $\operatorname{br} g$. Thus, $\sigma_p(T_{ug}) \subset B \cup K_2$.

Now suppose z is in K_1 and z is not zero. Fix n in C. It is our task to find f in l^2 such that $(T_{ug} - zI)f = 0$. Let $f_n = 1$, $f_{g^{(k)}(n)} = z^k (u(n) \cdots u(g^{(k-1)}(n)))^{-1}$ for $1 \leq k \leq c-1$, and $f_m = z^{-k} u(m) \cdots u(g^{(k)}(m)) f_{g^{(k+1)}(m)}$ for m in $\mathbf{Z}$ where $[g](m) = -k-1$. Define $\sum_{m\in\mathbf{Z}} f_m e_m = f$. Since g has finitely many branches and z is in $K_1\backslash\{0\}$, f is in l^2. Now

$$(T_{ug} - zI)f = \sum_{m\in\mathbf{Z}} (f_{g(m)} u(m) - z f_m) e_m = 0.$$

Thus, f is an eigenvector of T_{ug} for the eigenvalue z. Hence, $K_1 \subset \sigma_p(T_{ug})$.

There is a branch b of g such that $\operatorname{nl}(b) = -\infty$, since g has only one orbit and finitely many branches. If

$$f = \overline{u(g^{(c-1)}(b(0)))} e_{b(-1)} - u(b(-1)) e_{g^{(c-1)}(b(0))}$$

then $T_{ug}^* f = 0$. Thus, 0 is in the point spectrum of T_{ug}^*.

Let z in $\mathbf{C}$ be nonzero and $f = \sum_{n\in\mathbf{Z}} f_n e_n$ in l^2 be such that $f \neq 0$ and $(T_{ug}^* - zI)f = 0$. From (2.2) we have $z f_n - \sum_{k\in g^{-1}(n)} f_k u(k) = 0$ for all n in $\operatorname{Image} g$ and $f_n = 0$ for all n not in $\operatorname{Image} g$. Either there are two distinct branches a and b of g and $K > 0$ such that $\operatorname{nl}(a) = -\infty$, $\operatorname{nl}(b) = -\infty$,

$$f_{a(-k)} = z^k (u(a(-k)) \cdots u(a(-K-1)))^{-1} f_{a(-K)} \neq 0$$

and

$$f_{b(-k)} = z^k (u(b(-k)) \cdots u(b(-K-1)))^{-1} f_{b(-K)} \neq 0$$

for all $k > K$ or

$$f_{g^{(k)}(n)} = z^{-k} \overline{u(n)} \cdots \overline{u(g^{(k-1)}(n))} f_n \neq 0$$

for all n in $\mathbf{C}$ and $k > 0$. Thus, either $z^c = \overline{u(n)} \cdots \overline{u(g^{(c-1)}(n))}$ for n in $\mathbf{C}$ or for two distinct branches a and b of g such that $\mathrm{nl}(a) = -\infty$, $\mathrm{nl}(b) = -\infty$ $\{z^k(u(a(-1)) \cdots u(a(-k)))^{-1}\}_{k\geq 1}$ and $\{z^k(u(b(-1)) \cdots u(b(-k)))^{-1}\}_{k\geq 1}$ are square summable. Hence, z is in K_4.

Let z be in K_3 and $z \neq 0$. If $z^c = \overline{u(n)} \cdots \overline{u(g^{(c-1)}(n))}$ for n in $\mathbf{C}$ then fix n in $\mathbf{C}$ and let

$$f = \sum_{0\leq k\leq c-1} z^{-k}\overline{u(n)} \cdots \overline{u(g^{(k-1)}(n))} e_{g^{(k)}(n)}$$

if $|z| < L_a \leq L_b$ for two distinct, intersecting branches a and b of g such that $\mathrm{nl}(a) = -\infty$ and $\mathrm{nl}(b) = -\infty$. Then let

$$h = \sum_{k>|i|} \overline{u(b(i-1))} z^k (\overline{u(a(-k) \cdots u(a(i-2))})^{-1} e_{a(-k)} - \sum_{k>|i|} \overline{u(a(i-1))} z^k (\overline{u(b(-k)) \cdots u(b(i-2))})^{-1} e_{b(-k)},$$

where $i = \min[g](a \cap b)$. If $|z| < L_a \leq L_b$ for two nonintersecting branches a and b of g such that $\mathrm{nl}(a) = -\infty$, $\mathrm{nl}(b) = -\infty$, and $b(0) = g^{(t)}(a(0))$ then let

$$\begin{aligned} p = {} & \sum_{k>0} u(g^{(t-1)}(a(0))) z^k (u(b(-k)) \cdots u(b(-2)))^{-1} e_k \\ & - \sum_{0\leq k\leq t-1} u(b(-1)) z^{t-k} (u(g^{(k)}(a(0))) \cdots u(g^{(t-2)}(a(0)))^{-1} e_{g^{(k)}(a(0))} \\ & - \sum_{k>0} u(b(-1)) z^{t+k} (u(a(-k))) \cdots u(a(0)) \cdots u(g^{(t-2)}(a(0)))^{-1} e_{a(-k)}. \end{aligned}$$

The function f is in l^2 since f is a finite linear combination of basis elements. The functions h and p are in l^2 since $|z| < L_a \leq L_b$. Furthermore, $(T^*_{ug} - zI)f = 0$, $(T^*_{ug} - zI)h = 0$, and $(T^*_{ug} - zI)p = 0$ Hence, z is in the point spectrum of T^*_{ug}. □

Let Y be a subset of $\mathbf{Z}$, $g\colon Y \to \mathbf{Z}$ have finitely many branches and only one orbit, $\mathbf{Z} = Y \cup \mathrm{Image}\, g$, and u be a nonzero weight function. The operator T_{ug} is a finite rank perturbation of a finite direct sum of shift operators. We will make this more precise. There is an i in $\mathbf{Z}$ such that if i is in $[g]b$ for some branch b of g then $\mathrm{nl}(b) = -\infty$ and $\min[g](a \cap b) > i$ for any other branch a of g. There is a j in $\mathbf{Z}$ such that $[g]^{-1}(n)$ contains at most one element for $n \geq j$.

Let N be the span of $\{e_n \colon i < [g](n) \leq j\}$ and $F = (T_{ug}$ restricted to $N)$. Clearly, F has finite rank. Let $We_n = u(g^{-1}(n))e_{g^{(-1)}(n)}$ if $[g](n) > j$. For each a in $\mathrm{br}\, g$ such that $\mathrm{nl}(a) = -\infty$ let $W_a e_{a(k)} = u(a(k-1))e_{a(k-1)}$ if $k < i$. The spectra of W_a and W are well-studied sets (see Shields [25]). The operator $(T_{ug} - F)$ is equal to $W \oplus (\bigoplus_{a\in \mathrm{br}\, g,\, \mathrm{nl}(a)=-\infty} W_a)$. The following theorem is a consequence of standard results on spectra of compact perturbations.

Theorem 2.3. *Let Y, g, u, W, and W_a for a in* br g *be as above. Suppose T_{ug} is a bounded operator. Then*

(1) $\sigma_e(T_{ug}) = \sigma_e(W) \cup (\bigcup_{a\in \mathrm{br}\, g,\, \mathrm{nl}(a)=-\infty} \sigma_e(W_a))$,

(2) $\sigma(T_{ug}) = \sigma_p(T_{ug}) \cup \sigma_p(T^*_{ug}) \cup \sigma_e(T_{ug})$.

Example 5. Let $g\colon \mathbf{Z} \to \mathbf{Z}$ by

$$g(n) = \begin{cases} n+1 & \text{if } n \geq -1, \\ n+2 & \text{if } n < -1, \end{cases}$$

and let $u\colon \mathbf{Z} \to \mathbf{C}$ by

$$u(n) = \begin{cases} 1 & \text{if } n < -1 \text{ and } n = 2k \text{ for some } k \text{ in } \mathbf{Z}, \\ 2 & \text{if } n < -1 \text{ and } n = 2k-1 \text{ for some } k \text{ in } \mathbf{Z}, \\ 3 & \text{if } n \geq -1. \end{cases}$$

The function g has two branches, $a_1 = \{\dots, -4, -2, 0, 1, 2, \dots\}$ and $a_2 = \{\dots, -3, -1, 0, 1, 2, \dots\}$. The compact operator F is given by

$$Fe_n = \begin{cases} u(-2)e_{-2} + u(-1)e_{-1} & \text{if } n = 0, \\ u(n-2)e_{n-2} & \text{if } n = -1, -2, \\ 0 & \text{otherwise.} \end{cases}$$

Also,

$$We_n = u(n-1)e_{n-1} \quad \text{for } n \geq 1,$$
$$W_{a_1}e_n = u(n-2)e_{n-2} \quad \text{for } n < -2$$

and $n = 2k$ for some k in $\mathbf{Z}$, and

$$W_{a_2}e_n = u(n-2)e_{n-2} \quad \text{for } n < -1$$

and $n = 2k-1$ for some k in $\mathbf{Z}$. Now

$$\sigma_p(T_{ug}) = \{\mathbf{z} \text{ in } \mathbf{C}\colon 2 < |z| < 3\}, \qquad \sigma_p(T^*_{ug}) = \{z \text{ in } \mathbf{C}\colon |z| < 1\},$$
$$\sigma_e(W) = \{z \text{ in } \mathbf{C}\colon |z| = 3\}, \quad \sigma_e(W_{a_1}) = \{z \text{ in } \mathbf{C}\colon |z| = 1\},$$
$$\sigma_e(W_{a_2}) = \{z \text{ in } \mathbf{C}\colon |z| = 2\}.$$

Thus,

$$\sigma(T_{ug}) = \{z \text{ in } \mathbf{C}\colon |z| \leq 1\} \cup \{z \text{ in } \mathbf{C}\colon 2 \leq |z| \leq 3\}. \quad \square$$

We have seen that the structure of the spectrum of a weighted composition operator on l^2 is the type of structure we would get by taking direct sums of weighted shifts. Thus, from the perspective of weighted shifts the results given are not surprising. If we change our perspective and recall Example 4 of §1 and the paper of Cowen [7] then the results seem to be more unexpected. In Example 4, for a weighted composition operator T_g on H^2 of the unit disk there is an associated weighted composition operator $T_{ug^\sim}$ on l^2. Cowen [7] gives some results where $T_{ug^\sim}$ is a weighted shift whose spectrum can be an annulus but the spectrum of T_g is a disk. A natural question to ask is: Can we obtain results about the spectrum of T_g knowing the spectrum of $T_{ug^\sim}$? More particular, are there composition operators on H^2 of the disk whose spectra are the disjoint union of a disk and an annulus?

The hypothesis that the composition function g has finitely many branches was more than a minor assumption in this section. The results would be different if we were to assume g has infinitely many branches. Shields [25] makes note of an example of an operator with spectrum the unit disk, but the operator is the countably infinite direct sum of weighted shifts all of whose spectra are a single point. One of the difficulties in assuming g has infinitely many branches is that the operator F needed to make $T_{ug} - F$ a direct sum of weighted shifts may no longer be compact. Another difficulty would be in calculating the point spectrum of T_{ug} where the limits involved could move from branch to branch without progressing toward the end of any branch. The structure of $\sigma(T_{ug})$ is not clear when g has infinitely many branches.

3. Commutant

We wish to characterize the commutant of a weighted composition operator T_{ug} on l^2. In order to obtain the results given here it is necessary to make some nontrivial restrictions on the composition function g. These restrictions are: g is defined on all of $\mathbf{Z}$, g has exactly one orbit, g does not have a cycle, and all branches of g have the same negative length.

Consider Hilbert spaces of the form $L^2(\mathbf{Z}, \mu)$, where μ is a measure whose sigma-algebra is the power set of $\mathbf{Z}$. If μ is counting measure ($\mu\{n\} = 1$ for all n in $\mathbf{Z}$) then $l^2 = L^2(\mathbf{Z}, \mu)$. We are introducing weighted spaces $L^2(\mathbf{Z}, \mu)$ because an operator T_{ug} on l^2 is unitarily equivalent to an operator T_g on some weighted space $L^2(\mathbf{Z}, \mu)$. The computations are much easier when working with T_g on $L^2(\mathbf{Z}, \mu)$, hence, we will find the commutant of T_g on $L^2(\mathbf{Z}, \mu)$ instead of the commutant of T_{ug} on l^2. The subscript μ may be used to reference the space $L^2(\mathbf{Z}, \mu)$, for example, $\|\cdot\|_\mu$ will denote the norm on $L^2(\mathbf{Z}, \mu)$.

Let μ be a measure with sigma-algebra the power set of $\mathbf{Z}$. Then $\{e_n\}_{n\in\mathbf{Z}}$ is an orthogonal basis for $L^2(\mathbf{Z}, \mu)$, but $\{e_n\}_{n\in\mathbf{Z}}$ is not necessarily normalized since $\|e_n\|_\mu^2 = \mu(n)$ for n in $\mathbf{Z}$. For $f = \sum_{n\in\mathbf{Z}} f_n e_n$, $\|f\|_\mu^2 = \sum_{n\in\mathbf{Z}} |f_n|^2 \mu(n)$. If A is a linear operator on $L^2(\mathbf{Z}, \mu)$ then the matrix for A relative to $\{e_n\}_{n\in\mathbf{Z}}$ is given by $(A_{ij})_{i,j\in\mathbf{Z}}$, where $\mu(i)A_{ij} = \langle Ae_j, e_i\rangle_\mu$ for i and j in $\mathbf{Z}$. Weighted composition operators on a weighted space $L^2(\mathbf{Z}, \mu)$ are defined exactly as they were for the unweighted space l^2. The adjoint of a weighted composition operator T_{ug} on $L^2(\mathbf{Z}, \mu)$ does vary from the l^2 case; we present the formula for T_g^* acting on e_n, $n \in \mathbf{Z}$:

$$T_g^* e_n = \begin{cases} \frac{\mu(n)}{\mu(g(n))} e_{g(n)} & \text{for } n \text{ in the domain of } g, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 3.1. *Let Y be a subset of $\mathbf{Z}$, $g\colon Y \to \mathbf{Z}$ have one orbit and no cycle, $\mathbf{Z} = (Y \cup \operatorname{Image} g)$, and $u\colon \mathbf{Z} \to \mathbf{R}$ be a positive weight function. The operator T_{ug} on l^2 is unitarily equivalent to an operator T_g on $L^2(\mathbf{Z}, \mu)$ for some measure μ.*

Proof. Define the measure μ by $\mu(0) = 1$ and

$$\mu(n) = [u(0) \cdots u(g^{(t-1)}(0))[u(n) \cdots u(g^{(k-1)}(n))]^{-1}]^{-2},$$

where $g^{(k)}(n) = g^{(t)}(0)$ for some $k \geq 0$ and $t \geq 0$. Let $U\colon l^2 \to L^2(\mathbf{Z}, \mu)$ by $Uf = \mu^{-1/2} f$ for f in l^2. For f in l^2,

$$\|Uf\|_\mu^2 = \int_{\mathbf{Z}} \mu^{-1} |f|^2 \, d\mu = \int_{\mathbf{Z}} |f|^2 \, d1 = \|f\|^2.$$

If f is in l^2 and h is in $L^2(\mathbf{Z}, \mu)$ then

$$\begin{aligned} \langle Uf, h \rangle_\mu = \langle \mu^{-1/2} f, h \rangle_\mu &= \int_{\mathbf{Z}} \mu^{-1/2} f \bar{h} \, d\mu \\ &= \int_{\mathbf{Z}} f_\mu^{1/2} \bar{h} \, d1 = \langle f, \mu^{1/2} h \rangle = \langle f, U^{-1} h \rangle. \end{aligned}$$

Thus, $\|U\| = 1$ and $U^* = U^{-1}$ implying U is unitary. Let f be in l^2 and n be in $\mathbf{Z}$. Then

$$\begin{aligned} U^* T_g U f(n) &= U^* T_g((\mu^{-1/2} f)(n)) \\ &= \begin{cases} U^*((\mu \circ g)^{-1/2} (f \circ g)(n)) & \text{for } n \text{ in } Y, \\ 0 & \text{otherwise}, \end{cases} \\ &= \begin{cases} \mu^{1/2}(n) (\mu \circ g)^{-1/2}(n) (f \circ g)(n) & \text{for } n \text{ in } Y, \\ 0 & \text{otherwise}, \end{cases} \\ &= \begin{cases} \frac{u(0) \cdots u(g^{(t-1)}(0))[u(g(n)) \cdots u(g^{(k-2)}(g(n)))]^{-1}}{u(0) \cdots u(g^{(t-1)}(0))[u(n) \cdots u(g^{(k-1)}(n))]^{-1}} (f \circ g)(n) & \text{for } n \text{ in } Y, \\ \qquad \text{where } g^{(k)}(n) = g^{(t)}(0) \text{ for some } k \geq 0 \text{ and } t > 0, \\ 0 & \text{otherwise}, \end{cases} \\ &= \begin{cases} u(n)(f \circ g)(n) & \text{for } n \text{ in } Y, \\ 0 & \text{otherwise}, \end{cases} \\ &= T_{ug} f(n). \end{aligned}$$

Hence, T_g on $L^2(\mathbf{Z}, \mu)$ is unitarily equivalent to T_{ug} on l^2. □

Some of the material that follows requires us to look at operators that may not be bounded. In order to handle unboundedness we introduce the subspace $N(\mathbf{Z}, \mu)$ of $L^2(\mathbf{Z}, \mu)$ consisting of all finite linear combinations of the basis elements $\{e_n\}_{n\in\mathbf{Z}}$. If the composition function has a finite number of branches then a weighted composition operator and its adjoint both map $N(\mathbf{Z},\mu)$ into $N(\mathbf{Z},\mu)$. Every operator on $L^2(\mathbf{Z},\mu)$ has a natural restriction to an operator from $N(\mathbf{Z},\mu)$ into $L^2(\mathbf{Z},\mu)$ and every bounded operator on $N(\mathbf{Z},\mu)$ has a unique extension to an operator on $L^2(\mathbf{Z},\mu)$.

The following proposition will be used repeatedly during the rest of this exposition. It is also a generalization of Proposition 5 from [25].

Proposition 3.2. *Let μ be a positive measure on $\mathbf{Z}$ with sigma-algebra the power set of $\mathbf{Z}$ and let $g\colon \mathbf{Z}\to\mathbf{Z}$ be a function. Furthermore, let A be an operator from $N(\mathbf{Z},\mu)$ into $N(\mathbf{Z},\mu)$. The equation $AT_ge_n = T_gAe_n$ holds for all n in $\mathbf{Z}$ if and only if for all j and n in $\mathbf{Z}$,*

$$A_{g(j),n} = \begin{cases} \sum_{k\in g^{-1}(n)} A_{jk} & \text{if } n \text{ is in Image } g, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let n be in $\mathbf{Z}$. Then

$$T_gAe_n = T_g\left(\sum_{m\in\mathbf{Z}} A_{m,n}e_m\right) = \sum_{j\in\mathbf{Z}} A_{g(j),n}e_j$$

and

$$AT_ge_n = \begin{cases} A\left(\sum_{k\in g^{-1}(n)} e_k\right) & \text{if } n \text{ is in Image } g, \\ 0 & \text{otherwise,} \end{cases}$$
$$= \begin{cases} \sum_{j\in\mathbf{Z}}\sum_{k\in g^{-1}(n)} A_{jk}e_j & \text{if } n \text{ is in Image } g, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $T_gAe_n = AT_ge_n$ for n in $\mathbf{Z}$ if and only if, for all j and n in $\mathbf{Z}$,

$$A_{g(j),n} = \begin{cases} \sum_{k\in g^{-1}(n)} A_{jk} & \text{if } n \text{ is in Image } g, \\ 0 & \text{otherwise.} \quad \square \end{cases}$$

One case where the commutant of a weighted composition operator is well known is for weighted shifts. If T_g is a unilateral shift on a weighted space $L^2(\mathbf{Z},\mu)$ then T_g is unitarily equivalent to multiplication by z on a weighted H^2 space of the unit disc denoted $H^2(\beta)$. The commutant of multiplication by z on $H^2(\beta)$ is the set of operators that are multiplication by some h in $H^2(\beta)$ where hf is in $H^2(\beta)$ for all f in $H^2(\beta)$ (Shields [25]). A weighted composition operator in general is not multiplication by z on some space. Thus, we cannot hope for the same type of characterization of the commutant of a weighted composition operator that is not a shift. Every h in $H^2(\beta)$ is a formal power series in z and so commutes with z. We will pursue this idea and try to characterize the commutant of T_g on $L^2(\mathbf{Z},\mu)$ as the set of operators T_h where h commutes with g.

In Proposition 3.3 we show that to determine the structure of an operator A in the commutant of T_g on $L^2(\mathbf{Z},\mu)$ it is sufficient to determine the structure of one diagonal of finite dimensional submatrices for the matrix of A.

Theorem 3.4 treats the case when only one of the submatrices of the diagonal of A determined in Proposition 3.3 is nonzero. The results of Theorem 3.4 are extended in Theorem 3.5 where all operators in the commutant of T_g are characterized.

Let $g\colon \mathbf{Z} \to \mathbf{Z}$ have one orbit, no cycle and finitely many branches, and let μ be a positive measure on $\mathbf{Z}$. Define $F[g] = \sup\{n\colon g^{-1}(k)$ contains at most one element for all k such that $[g](k) \leq n\}$. For m in $\mathbf{Z}$ and b a branch of g define $S_m^g = \operatorname{span}\{e_n\colon [g](n) = m\}$ and

$$m_b = \begin{cases} \max(F[g] + m, \operatorname{nl} b) & \text{if } m < 0, \\ \max(F[g], \operatorname{nl} b) & \text{if } m \geq 0. \end{cases}$$

Proposition 3.3. *Let A and B be operators on $N(\mathbf{Z}, \mu)$ with matrices $(A_{ij})_{i,j\in\mathbf{Z}}$ and $(B_{ij})_{i,j\in\mathbf{Z}}$. Let $g\colon \mathbf{Z} \to \mathbf{Z}$ have more than one branch, finitely many branches, no cycles and one orbit. Furthermore, suppose that $T_gAe_n = AT_ge_n$ and $T_gBe_n = BT_ge_n$ for all $n \in \mathbf{Z}$. Then $A_{ij} = B_{ij}$ for all i and j in $\mathbf{Z}$ if and only if $A_{b(m_b),\beta(m_b-m)} = B_{b(m_b),\beta(m_b-m)}$ for all m in $\mathbf{Z}$ and b, $\beta \in \operatorname{br} g$ such that $\operatorname{nl}\beta \leq m_b - m$.*

Proof. It is clear that $A_{ij} = B_{ij}$ for all i and j in $\mathbf{Z}$ implies $A_{b(m_b),\beta(m_b-m)} = B_{b(m_b),\beta(m_b-m)}$ for all m in $\mathbf{Z}$ and b and β in $\operatorname{br} g$ such that $\operatorname{nl}\beta \leq m_b - m$.

Conversely, let i and j be in $\mathbf{Z}$. Choose $s = [g](i)$, $t = [g](j)$ and $m = s - t$. If $i = b(s)$ and $s = m_b$ for some b in $\operatorname{br} g$ then $j = \beta(t) = \beta(m_b - m)$ for some β in $\operatorname{br} g$ and $A_{ij} = B_{ij}$.

Suppose $i = b(s)$ and $s < m_b$ for some b in $\operatorname{br} g$. Now $m_b - (s-t) = m_b - m \leq F[g]$ and, by Proposition 3.2, $B_{h(s+x),\beta(t+x)} = B_{h(s+x+1),\beta(t+x+1)}$ for all $0 \leq x \leq m_b - s - 1$ and h, β in $\operatorname{br} g$ such that $h(s+x)$ and $\beta(t+x)$ exist. Thus

$$B_{ij} = B_{b(s),\beta(t)} = B_{b(s+(m_b-s-1)+1),\beta(t+(m_b-s-1)+1)} = B_{b(m_b),\beta(m_b-m)}$$

where $j = \beta(t)$ for some β.

Finally, suppose $s > m_b$ for all b in $\operatorname{br} g$ such that $i = b(s)$. If b and β are in $\operatorname{br} g$ such that $i = b(s)$, $j = \beta(t)$ and $\operatorname{nl}\beta \leq m_b - m$ then

$$B_{b(s-(s-m_b)),\beta(t-(s-m_b))} = B_{b(m_b),\beta(m_b-m)}.$$

If b and β are in $\operatorname{br} g$ such that $i = b(s)$, $j = \beta(t)$ and $\operatorname{nl}\beta > m_b - m$ then

$$B_{b(s-(t-\operatorname{nl}\beta)),\beta(t-(t-\operatorname{nl}\beta))} = B_{b(s-t+\operatorname{nl}\beta),\beta(\operatorname{nl}\beta)} = 0$$

by Proposition 3.2.

Now let b and β be in $\operatorname{br} g$ such that $i = b(s)$ and $j = \beta(t)$, let $0 \le x < s - m_b$ and let $\operatorname{nl} \beta \le t - x$. Then, by Proposition 3.2,

$$B_{b(s-x),\beta(t-x)} = \sum_{y \in g^{-1}(\beta(t-x))} B_{b(s-x-1),y}$$

$$= \sum_{\substack{h \in \operatorname{br} g \\ h(t-s)=\beta(t-x) \\ \operatorname{nl} h \le t-x-1}} \frac{1}{H_{h,t-x-1}} B_{b(s-x-1),h(t-x-1)}$$

(where $H_{h,t-x-1} = \text{cardinality}\{a \text{ in } \operatorname{br} g \colon a(t-x-1) = h(t-x-1)\}$)

$$= \sum_{\substack{\beta \in \operatorname{br} g \\ \beta(t)=j \\ \operatorname{nl} \beta \le m_b - m}} \frac{1}{M_\beta} B_{b(m_b),\beta(m_b-m)},$$

where b is in $\operatorname{br} g$ with $b(s) = i$ and $M_\beta = \prod_{i=m_b-m}^{t-x-1} H_{\beta,i}$.

We will now characterize those operators A that commute with T_g. The first result, Theorem 3.4, addresses the case when A is an operator weighted shift acting on the subspaces S_m^g. Theorem 3.4 characterizes A as a linear combination of operators T_x where x commutes with g. The second result, Theorem 3.5, extends the result of Theorem 3.4 to all operators that commute with T_g.

Theorem 3.4. *Let $g \colon \mathbf{Z} \to \mathbf{Z}$ have one orbit, no cycle, more than one branch, and only a finite number of branches. In addition, assume there is a K in $\mathbf{Z} \cup \{-\infty\}$ such that $\operatorname{nl} b = K$ for all b in $\operatorname{br} g$. On the space $L^2(\mathbf{Z}, \mu)$, if A is in the commutant of T_g and there is an m in $\mathbf{Z}$ such that $A(S_p^g) \subset (S_{p+m}^g)$ for all p in $\mathbf{Z}$ then A restricted to $N(\mathbf{Z}, \mu)$ is in the span of $M = \{T_X \colon X \circ g = g \circ X, X \colon \mathbf{Z} \to \mathbf{Z}\}$.*

Proof. Let $\operatorname{br} g = \{b_1, \dots, b_n\}$, $\mathbf{J} = \{1, \dots, n\}$, m be a fixed integer and A be in the commutant of T_g such that $A(S_p^g) \subset (S_{p+m}^g)$ for all p in $\mathbf{Z}$. The matrix for A with respect to $\{e_n\}$ will be denoted $(A_{ij})_{i,j \in \mathbf{Z}}$.

Since all branches of g have the same negative length there is a fixed integer $P = m_b$ for all b in $\operatorname{br} g$. We need to show that the matrix for A is in the span of M. By Proposition 3.3, we need to show there is a matrix $(B_{ij})_{i,j \in \mathbf{Z}}$ in the span of M with $A_{b(P),\beta(P-m)} = B_{b(P),\beta(P-m)}$ for all b, β in $\operatorname{br} g$ such that $\operatorname{nl} \beta \le P - m$, since it is clear that $A_{b(s),\beta(t)} = 0$ for all b, β in $\operatorname{br} g$ and s, t in $\mathbf{Z}$ such that $s - t \ne m$. For each k in $\mathbf{Z}$ and p in $\mathbf{J}$ define $\varsigma_{pk} = \{s \text{ in } \mathbf{J} \colon \inf[g](b_s \cap b_p) \le F[g] + k\}$.

Note that if $k \le 0$ then

$$\varsigma_{pk} = \begin{cases} \varnothing & \text{if } F[g] - K < -k, \\ \{p\} & \text{if } F[g] - K \ge -k, \end{cases}$$

where $K = \operatorname{nl} b$ for all b in $\operatorname{br} g$.

We divide the proof into cases depending on the values of m and K.

Case 1. The value K is finite and the integer m is strictly positive. By Proposition 3.2 we have

$$0 = A_{b(\text{nl}\,\beta+m),\beta(\text{nl}\,\beta)} = A_{b(\text{nl}\,\beta+m+1),\beta(\text{nl}\,\beta+1)} = \cdots = A_{b(P),\beta(P-m)}$$

for all b and β in $\text{br}\,g$ such that $\text{nl}\,\beta \leq P - m$. Thus, $A \equiv 0$ and A is in the span of M.

Case 2. The integer m is less than or equal to zero, or K is infinite and m is positive. In this case $\text{nl}\,\beta \leq P - m$ for all b and β in $\text{br}\,g$ and the set of entries $A_{b(P),\beta(P-m)}$ can be index by $\mathbf{J} \times \mathbf{J}$. For each (s,t) in $\mathbf{J} \times \mathbf{J}$ define $\lambda_{\text{st}}: \mathbf{Z} \to \mathbf{Z}$ by

$$\lambda_{\text{st}}(b_y(x)) = \begin{cases} b_z(x-m) & \text{if } y \neq s \text{ where } p = [\min[g](b_y \cap b_s)] - F[g] \\ & \quad \text{and } z = \min ç_{t,p-m}, \\ b_t(x-m) & \text{if } y = s, \end{cases}$$

for y in $\mathbf{J}$ and x in $\mathbf{Z}$. The following two lemmas show that $T_{\lambda_{\text{st}}}$ is in M for all (s,t) in $\mathbf{J} \times \mathbf{J}$.

Lemma 1. *For each* (s,t) *in* $\mathbf{J} \times \mathbf{J}$ *the function* λ_{st} *is well defined.*

Proof. Suppose (s,t) is in $\mathbf{J} \times \mathbf{J}$, y_1 is in $\mathbf{J}$, y_2 is in $\mathbf{J}$ and x is in $\mathbf{Z}$ such that $b_{y_1}(x) = b_{y_2}(x)$. We need to show that $\lambda_{\text{st}}(b_{y_1}(x)) = \lambda_{\text{st}}(b_{y_2}(x))$. Let $L_1 = [\min[g](b_{y_1} \cap b_s)] - F[g]$, $L_2 = [\min[g](b_{y_2} \cap b_s)] - F[g]$, $z_1 = \min ç_{t,L_1-m}$ and $z_2 = \min ç_{t,L_2-m}$. If $x < L_1 + F[g]$ and $x < L_2 + F[g]$ then

$$b_s(L_1 + F[g]) = b_{y_1}(L_1 + F[g]) = b_{y_2}(L_1 + F[g])$$

and

$$b_s(L_2 + F[g]) = b_{y_2}(L_2 + F[g]) = b_{y_1}(L_2 + F[g])$$

so $L_1 + F[g] \geq L_2 + F[g]$ and $L_1 + F[g] \leq L_2 + [g]$ implying $L_1 = L_2$ and $z_1 = z_2$; thus,

$$\lambda_{\text{st}}(b_{y_1}(x)) = b_{z_1}(x-m) = b_{z_2}(x-m) = \lambda_{\text{st}}(b_{y_2}(x)).$$

If $x \geq L_1 + F[g]$ or $x \geq L_2 + F[g]$ then $b_{y_1}(x) = b_s(x) = b_{y_2}(x)$ so $x \geq L_1 + F[g]$ and $x \geq L_2 + F[g]$ implying $x - m \geq L_1 - m + F[g]$ and $x - m \geq L_2 - m + F[g]$; thus, z_1 and z_2 are in $ç_{t,x-m}$ and

$$\lambda_{\text{st}}(b_{y_1}(x)) = b_{z_1}(x-m) = b_{z_2}(x-m) = \lambda_{\text{st}}(b_{y_2}(x)).$$

Lemma 2. *For each* (s,t) *in* $\mathbf{J} \times \mathbf{J}$ $\lambda_{\text{st}} \circ g = g \circ \lambda_{\text{st}}$.

Proof. Let k be in $\mathbf{Z}$; then $k = b_y(x)$ for some y in $\mathbf{J}$ and x in $\mathbf{Z}$. Now

$$\begin{aligned} \lambda_{\text{st}} \circ g(k) &= \lambda_{\text{st}}(g(b_y(x))) = \lambda_{\text{st}}(b_y(x+1)) = b_z(x+1-m) \\ &= g(b_z(x-m)) = g(\lambda_{\text{st}}(b_y(x))) = g \circ \lambda_{\text{st}}(k), \end{aligned}$$

where $z = \min ç_{t,p-m}$ and $p = [\min[g](b_y \cap b_s)] - F[g]$.

In view of Lemmas 1 and 2 we need only show A is a linear combination of operators of the form $T_{\lambda_{st}}$ where (s, t) is in $\mathbf{J}\times\mathbf{J}$. To accomplish this, partition $\mathbf{J}$ into the subsets $O_k = \{t \text{ in } \mathbf{J}\colon t \neq \min \varsigma_{t,-m+k-1} \text{ and } t = \min \varsigma_{t,-m+k}\}$ for $k_1 \leq k < k_2$, where $k_1 = \max(0, m)$, $k_2 = \max(h, h+m)$ where $h = \min\{q \text{ in } \mathbf{Z}\colon [g](q) = [g](r) \text{ implies } q = r\}$; $O_{k_2} = \{t \text{ in } \mathbf{J}\colon t = \min \varsigma_{t,-m+k_2}\}$ and $O_{k_1-1} = \varnothing$. Let $B_{k_1-1} = A$ and, for $k_1 \leq k \leq k_2$, let

$$B_k = B_{k-1} - \sum_{\substack{t\in O_k \\ s=\min \varsigma_{s,k}}} (B_{k-1})_{b_s(P), b_t(P-m)} T_{\lambda_{st}}.$$

We now need to show that B_{k_2} is the zero operator; then A will clearly be in the span of M. By virtue of Proposition 3.3, this amounts to showing that $(B_{k_2})_{b_s(P), b_t(P-m)} = 0$ for all (s, t) in $\mathbf{J}\times\mathbf{J}$.

We proceed inductively. Our inductive hypothesis is

$$(B_k)_{b_q(P), b_r(P-m)} = 0 \quad \text{if } q \in \mathbf{J} \quad \text{and} \quad r \in \bigcup_{i=k_1-1}^{k} O_i.$$

The induction hypothesis is trivially true for $k = k_1 - 1$ since $\bigcup_{i=k_1-1}^{k_1-1} O_i$ is the empty set.

Now suppose that for some k, where $k_1 - 1 \leq k < k_2$, $(B_k)_{b_q(P), b_r(P-m)} = 0$ when $q \in \mathbf{J}$ and $r \in \bigcup_{i=k_1-1}^{k} O_i$. We will demonstrate that $(B_{k+1})_{b_q(P), b_r(P-m)} = 0$ when $q \in \mathbf{J}$ and $r \in \bigcup_{i=k_1-1}^{k+1} O_i$.

If t in $\mathbf{J}$ such that $t = \min \varsigma_{t,-m+k+1}$, s in $\mathbf{J}$ such that $s = \min \varsigma_{s,k+1}$ and q in $\varsigma_{s,k+1}$ then using Proposition 3.2 and the inductive hypothesis we see that

$$\begin{aligned}
(B_k)_{b_q(P), b_t(P-m)} &= (B_k)_{b_q(P), b_t(P-m)} + 0 \\
&= (B_k)_{b_q(P), b_t(P-m)} + \sum_{\substack{r\in\varsigma_{t,-m+k+1} \\ r\neq t}} (B_k)_{b_q(P), b_r(P-m)} \\
&= \sum_{r\in\varsigma_{t,-m+k+1}} (B_k)_{b_q(P), b_r(P-m)} = \sum_{r\in\varsigma_{t,-m+k+1}} (B_k)_{b_q(P+1), b_r(P-m+1)} \\
&= \cdots = \sum_{r\in\varsigma_{t,-m+k+1}} (B_k)_{b_q(F[g]+m), b_r(F[g])} \\
&= \sum_{\substack{r\in\varsigma_{t,-m+k+1} \\ r=\min \varsigma_{r,1}}} (B_k)_{b_q(F[g]+m+1), b_r(F[g]+1)} \\
&= \sum_{\substack{r\in\varsigma_{t,-m+k+1} \\ r=\min \varsigma_{r,2}}} (B_k)_{b_q(F[g]+m+2), b_r(F[g]+2)} \\
&= \cdots = (B_k)_{b_q(F[g]+k+1), b_t(F[g]-m+k+1)}
\end{aligned}$$

since $t = \min \varsigma_{t,-m+k+1}$. Hence, $(B_k)_{b_q(P), b_r(P-m)} = (B_k)_{b_x(P), b_t(P-m)}$ for $s = \min \varsigma_{s,k+1}$, q and x in $\varsigma_{s,k+1}$ and t in O_{k+1}.

Let $q, r, s, t \in \mathbf{J}$ and $y = \min[g](b_q \cap b_s) - F[g]$. Then

$$\begin{aligned}
&(T_{\lambda_{st}})_{b_q(P),b_r(P-m)} \\
&\quad = \frac{1}{\mu(b_q(P))}\langle T_{\lambda_{st}} e_{b_r(P-m)}, e_{b_q(P)}\rangle \\
&\quad = \frac{1}{\mu(b_q(P))}\langle e_{b_r(P-m)}, T^*_{\lambda_{st}} e_{b_q(P)}\rangle \\
&\quad = \frac{1}{\mu(\lambda_{st}(b_q(P)))}\langle e_{b_r(P-m)}, e_{\lambda_{st}(b_q(P))}\rangle \\
&\quad = \begin{cases} \frac{1}{\mu(b_t(P-m))}\langle e_{b_r(P-m)}, e_{b_t(P-m)}\rangle & \text{if } q = s, \\ \frac{1}{\mu(b_z(P-m))}\langle e_{b_r(P-m)}, e_{b_z(P-m)}\rangle & \text{if } q \neq s, \text{ where } z = \min \varsigma_{t,y-m}, \end{cases} \\
&\quad = \begin{cases} 0 & \text{if } q = s \text{ and } r \neq t, \\ 1 & \text{if } q = s \text{ and } r = t, \\ 0 & \text{if } q \neq s \text{ and } r \neq \min \varsigma_{t,y-m}, \\ 1 & \text{if } q \neq s \text{ and } r = \min \varsigma_{t,y-m}. \end{cases}
\end{aligned}$$

Thus,

$$\begin{aligned}
&(B_k)_{b_s(P),b_t(P-m)}(T_{\lambda_{st}})_{b_q(P),b_r(P-m)} \\
&\qquad = \begin{cases} (B_k)_{b_s(P),b_t(P-m)} & \text{if } (q,r) = (s,t) \text{ or } q \neq s \\ & \text{and } r = \min \varsigma_{t,y-m}, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

If $r \in \bigcup_{i=k_1-1}^{k} O_i$ then $r \neq t$ and $r \neq \min \varsigma_{t,y-m}$, since $t \in O_{k+1}$. So

$$(B_k)_{b_s(P),b_t(P-m)}(T_{\lambda_{st}})_{b_q(P),b_r(P-m)} = 0.$$

If $r \in O_{k+1}$ then

$$\begin{aligned}
&(B_k)_{b_s(P),b_t(P-m)}(T_{\lambda_{st}})_{b_q(P),b_r(P-m)} \\
&\qquad = \begin{cases} (B_k)_{b_q(P),b_r(P-m)} & \text{if } r = t \text{ and } q = s, \\ (B_k)_{b_s(P),b_t(P-m)} & \text{if } r = t, s = \min \varsigma_{q,k+1} \neq q, \\ 0 & \text{otherwise,} \end{cases} \\
&\qquad = \begin{cases} (B_k)_{b_q(P),b_r(P-m)} & \text{if } r = t \text{ and } s = \min \varsigma_{q,k+1}, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Let r be in $\bigcup_{i=k_1-1}^{k} O_i$ and q be in $\mathbf{J}$. Then

$$\left(B_k - \sum_{\substack{t \in O_{k+1} \\ s = \min \varsigma_{s,k+1}}} (B_k)_{b_s(P),b_t(P-m)} T_{\lambda_{st}} \right)_{b_q(P),b_r(P-m)} = (B_k)_{b_q(P),b_r(P-m)} - 0 = 0$$

TABLE 1

Spaces		$\cdots$	S^g_{-3}	S^g_{-2}	S^g_{-1}	S^g_0	S^g_1	$\cdots$
	Basis Elements	$\cdots$	e_{-6} e_{-5}	e_{-4} e_{-3}	e_{-2} e_{-1}	e_0	e_1	$\cdots$
$\vdots$	$\vdots$	$\ddots$						
S^g_{-2}	e_{-4} e_{-3}		A_{-4-6} A_{-4-5} A_{-3-6} A_{-3-5}	A_{-4-4} A_{-4-3} A_{-3-4} A_{-3-3}	A_{-4-2} A_{-4-1} A_{-3-2} A_{-3-1}	A_{-40} A_{-30}	A_{-41} A_{-31}	
S^g_{-1}	e_{-2} e_{-1}		A_{-2-6} A_{-2-5} A_{-1-6} A_{-1-5}	A_{-2-4} A_{-2-3} A_{-1-4} A_{-1-3}	A_{-2-2} A_{-2-1} A_{-1-2} A_{-1-1}	A_{-20} A_{-1}	A_{-21} A_{-11}	
S^g_0	e_0		A_{0-6} A_{0-5}	A_{0-4} A_{0-3}	A_{0-2} A_{0-1}	A_{00}	A_{01}	
S^g_1	e_1		A_{1-6} A_{1-5}	A_{1-4} A_{1-3}	A_{1-2} A_{1-1}	A_{10}	A_{11}	
$\vdots$	$\vdots$						$\ddots$	

by the induction hypothesis. Let r be in O_{k+1} and q be in $\mathbf{J}$. Then

$$\left(B_k - \sum_{\substack{t\in O_{k+1} \\ s=\min \varsigma_{s,k+1}}} (B_k)_{b_s(P),b_t(P-m)} T_{\lambda_{st}} \right)_{b_q(P),b_r(P-m)}$$

$$= (B_k)_{b_q(P),b_r(P-m)} - \sum_{\substack{t=r \\ s=\min \varsigma_{q,k+1}}} (B_k)_{b_s(P),b_t(P-m)}$$

$$= (B_k)_{b_q(P),b_r(P-m)} - (B_k)_{b_q(P),b_r(P-m)} = 0.$$

Hence,

$$(B_{k+1})_{b_q(P),b_r(P-m)}$$

$$= \left(B_k - \sum_{\substack{t\in O_{k+1} \\ s=\min \varsigma_{s,k+1}}} (B_k)_{b_s(P),b_s(P-m)} T_{\lambda_{st}} \right)_{b_q(P),b_r(P-m)} = 0$$

if r is in $\bigcup_{i-k_1-1}^{k+1} O_i$ and q is in $\mathbf{J}$. Therefore $(B_{k_2})_{b_q(P),b_r(P-m)} = 0$ for all q and r in $\mathbf{J}$. □

The results of Theorem 3.4 can be extended to all operators in the commutant of T_g. Denote the set of bounded operators on $L^2(\mathbf{Z},\mu)$ by $B(L^2(\mathbf{Z},\mu))$. For A in $B(L^2(\mathbf{Z},\mu))$ with matrix $(A_{ij})_{i,j\in\mathbf{Z}}$ and m in $\mathbf{Z}$ define $D(A,m)$ on $L^2(\mathbf{Z},\mu)$ to be the operator with matrix $(D(A,m)_{i,j})_{i,j\in\mathbf{Z}}$ given by

$$D(A,m)_{i,j} = \begin{cases} A_{ij} & \text{if } [g](i) = [g](j) + m, \\ 0 & \text{otherwise.} \end{cases}$$

TABLE 2

Spaces		$\cdots$	S^g_{-3}	S^g_{-2}	S^g_{-1}	S^g_0	S^g_1	$\cdots$
	Basis Elements	$\cdots$	$e_{-6}\ e_{-5}$	$e_{-4}\ e_{-3}$	$e_{-2}\ e_{-1}$	e_0	e_1	$\cdots$
$\vdots$	$\vdots$	$\ddots$						
S^g_{-2}	e_{-4} e_{-3}		$A_{-4-6}\ A_{-4-5}$ $A_{-3-6}\ A_{-3-5}$	0 0 0 0	0 0 0 0	0 0	0 0	
S^g_{-1}	e_{-2} e_{-1}		0 0 0 0	$A_{-2-4}\ A_{-2-3}$ $A_{-1-4}\ A_{-1-3}$	0 0 0 0	0 0	0 0	
S^g_0	e_0		0 0	0 0	$A_{0-2}\ A_{0-1}$	0	0	
S^g_1	e_1		0 0	0 0	0 0	A_{10}	0	
$\vdots$	$\vdots$						$\ddots$	

The operators $D(A,m)$ are in $B(L^2(\mathbf{Z},\mu))$ and A is the weak operator limit of $A_n = \sum_{m=-n}^{n} D(A,m)$ as n goes to ∞.

Example 6. Let $g\colon \mathbf{Z} \to \mathbf{Z}$ be the function that has the graph:

$$\begin{array}{ccc} \cdots \to -3 \to -1 & & \\ & \searrow & \\ & & 0 \to 1 \to \cdots \\ & \nearrow & \\ \cdots \to -4 \to -2 & & \end{array}$$

Without loss of generality, let $[g](0) = 0$. Then $S^g_m = \operatorname{span}\{e_{-2m}, e_{-2m+1}\}$ when $m < 0$ and $S^g_m = \operatorname{span}\{e_m\}$ and $m \geq 0$. Suppose that A is an operator that commutes with T_g. Table 1 illustrates the matrix for A indexed in relation to the spaces S^g_m and also to the basis elements e_m.

An operator $D(A,m)$ is an operator that has zero entries except for one matrix diagonal (indexed by the spaces S_i). Table 2 illustrates $D(A,1)$. Note that

$$\begin{pmatrix} A_{-2,-4} & A_{-2,-3} \\ A_{-1,-4} & A_{-1,-3} \end{pmatrix}$$

are the matrix entries in Proposition 3.3.

Theorem 3.5. *Let $g\colon \mathbf{Z} \to \mathbf{Z}$ have one orbit, no cycle, more than one branch, and only a finite number of branches. In addition assume there is K in $\mathbf{Z} \cup \{-\infty\}$ such that $\operatorname{nl} b = k$ for all b in $\operatorname{br} g$. Then in the space $L^2(\mathbf{Z},\mu)$,*

$$\{T_g\}' = [B(L^2(\mathbf{Z},\mu)) \cap \operatorname{span}\{T_x : X \circ g = g \circ X, X : \mathbf{Z} \to \mathbf{Z}\}].$$

Proof. Let A be in $\{T_g\}'$ and m be in $\mathbf{Z}$. The operator $D(A,m)$ satisfies Proposition 3.2 and therefore $D(A,m)$ is in $\{T_g\}'$. Theorem 3.4 implies $D(A,m)$ is in the span of $\{T_x\colon X \circ g = g \circ X, X\colon \mathbf{Z} \to \mathbf{Z}\}$. Hence, for any positive integer n, A_n is in the span of $\{T_x\colon X \circ g = g \circ X, X\colon \mathbf{Z} \to \mathbf{Z}\}$. Conversely, it is clear that $B(L^2(\mathbf{Z},\mu)) \cap \operatorname{span}\{T_x\colon X \circ g = g \circ X, X\colon \mathbf{Z} \to \mathbf{Z}\}$ is contained in $\{T_g\}'$. □

Theorem 3.5 leaves us with some unanswered questions. Is it possible to choose functions that commute with T_g in such a way that the commutant of T_g is the weak closure of the span of $\{T_x\colon X \circ g = g \circ X$ and T_x is bounded$\}$ or maybe even that the commutant of T_g is the span of $\{T_x\colon X \circ g = g \circ X$ and T_x is bounded$\}$? What should the conclusion of Theorem 3.5 be if the branches of g are allowed to have different negative lengths? In this case the functions λ_{st} constructed in Theorem 3.4 may not commute with g, so that an easy modification of Theorem 3.4 is not possible.

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